Hankel Determinant for a Subclass of Alpha Convex Functions

Gagandeep Singh and Gurcharanjit Singh

Abstract--- In the present investigation, the upper bound of second Hankel determinant $|a_2 a_4 - a_3^2|$ for functions belonging to the subclass $M(\alpha; A, B)$ of analytic functions is studied. Results presented in this paper would extend the corresponding results of various authors.

Keywords--- Analytic Functions, Starlike Functions, Convex Functions, Alpha Convex Functions, Subordination, Schwarz Function, Second Hankel Determinant

I. INTRODUCTION

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

In the unit disc $E = \{z : |z| < 1\}$.

By S we denote the class of functions $f(z) \in A$ and univalent in E .

Let U be the class of Schwarzian functions

$$w(z) = \sum_{k=1}^{\infty} d_k z^k$$

Which are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$.

Let f and g be two analytic functions in E . Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarz function $w(z) \in U$, such that $f(z) = g(w(z))$.

In 1976, Noonan and Thomas [9] stated the q th Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinant plays an important role in the study of singularities; for instance, see [2]. This is also important in the study of power series with integral coefficients, see [2] and Cantor [1]. Hankel determinants are useful in showing that a function of bounded characteristic in the unit disc i.e. a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational. It is well known that the Fekete-Szegő functional

$$|a_3 - a_2^2| = H_2(1).$$

This functional is further generalized as $|a_3 - \mu a_2^2|$ for μ (real as well as complex). It is a very great combination of the two coefficients which describes the area problems posted earlier by Gronwall in 1914-15. Moreover, we also know that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $H_2(2)$. The functional $|a_2 a_4 - a_3^2|$ has been studied by many authors see ([4-6], [10], [14-16]). Janteng et al. [4] have considered the functional $|a_2 a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass R of S , consisting of functions whose derivative has a positive real part studied by MacGregor [9]. In their work, they have shown that if $f \in R$, then $|a_2 a_4 - a_3^2| \leq \frac{4}{9}$. The same authors [5] also obtained

the second Hankel determinant and sharp upper bounds for the familiar subclasses namely, starlike and convex functions denoted by S^* and K of S and have shown that $|a_2 a_4 - a_3^2| \leq 1$ and $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$ respectively.

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we introduce certain subclass of analytic functions and obtain an upper bound to the functional $|a_2 a_4 - a_3^2|$ for the function f belonging to this class, defined as follows:

$M(\alpha; A, B)$ denote the subclass of functions $f(z) \in A$ and satisfying the condition

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$$(1-\alpha)\frac{f'(z)}{f(z)} + \alpha\frac{(f'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, 0 \leq \alpha \leq 1, z \in E \quad (1.2)$$

The class $M(\alpha; A, B)$ is the subclass of alpha-convex functions. The following observations are obvious:

- $M(\alpha; 1, -1) \equiv M(\alpha)$, the class introduced by Mocanu[11].
- $M(0; 1, -1) \equiv S^*$, the class of starlike functions.
- $M(1; 1, -1) \equiv K$, the class of convex functions.
- $M(0; A, B) \equiv S^*(A, B)$, the subclass of starlike functions studied by Goel and Mehrotra [3].
- $M(1; A, B) \equiv K(A, B)$, the subclass of convex functions studied by Goel and Mehrotra [3].

Results due to Janteng et al.[5] and Singh and Singh [16] follows as special cases from the results of this paper.

II. PRELIMINARY RESULTS

Let P be the family of all functions p analytic in E for which $\operatorname{Re}(p(z)) > 0$ and

Where

$$P(\alpha) = - \left\{ \begin{aligned} & \left\{ 6(1+5\alpha)(1+3\alpha)(1+2\alpha) - 4(1+2\alpha)^2(1+7\alpha) - 3(1+3\alpha)^3 \right\} A^2 \\ & + \left\{ 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) + 6(1+5\alpha)(1+3\alpha)(1+2\alpha) \right. \\ & \quad \left. - 4(1+\alpha)^3(1+2\alpha)^2 - 4(1+2\alpha)^2(1+7\alpha) \right. \\ & \quad \left. - 3(1+3\alpha)^3 - 3(1+\alpha)^4(1+3\alpha) - 6(1+3\alpha)^2(1+\alpha)^2 \right\} B^2 \\ & + \left\{ 6(1+3\alpha)^3 + 6(1+3\alpha)^2(1+\alpha)^2 - 12(1+5\alpha)(1+3\alpha)(1+2\alpha) \right\} AB \\ & \quad \left. - 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) + 8(1+2\alpha)^2(1+7\alpha) \right\} \end{aligned} \right| \quad (3.2)$$

And

$$Q(\alpha) = 4 \left| \begin{aligned} & 6\alpha(1+\alpha)^3 A \\ & + 2(1+\alpha)^2 \left\{ 3(1+3\alpha)(2+5\alpha+\alpha^2) - (1+2\alpha)(7+27\alpha+8\alpha^2) \right\} B \\ & - 8(1+\alpha)^3 [1+4\alpha+\alpha^2] \end{aligned} \right| \quad (3.3)$$

Proof: If $f(z) \in M(\alpha; A, B)$, then there exists a Schwarz function $w(z) \in U$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = \phi(w(z)) \quad (3.4)$$

Where

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + \dots$$

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

for $z \in E$.

Lemma 2.1: [13] If $p \in P$, then $|p_k| \leq 2$ ($k = 1, 2, 3, \dots$).

Lemma 2.2: [7,8] If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

For some x and z Satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

III. MAIN RESULT

Theorem 3.1: If $f \in M(\alpha; A, B)$, then

$$\left| a_2 a_3 - a_3^2 \right| \leq \frac{(A-B)^2}{192(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)} \left[\frac{(Q(\alpha))^2}{4P(\alpha)} + 48(1+\alpha)^4(1+3\alpha) \right] \quad (3.1)$$

$$= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad (3.5)$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots \quad (3.6)$$

Since $w(z)$ is a Schwarz function, we see that $\operatorname{Re}(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $h(z)$ by

$$h(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \dots \quad (3.7)$$

In view of the equations (3.4), (3.6) and (3.7), we have

$$\begin{aligned}
h(z) &= \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1 z + c_2 z^2 + c_3 z^3 + \dots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \dots}\right) \\
&= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \dots\right) \\
&= 1 + \frac{B_1 c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]z^2 \\
&\quad + \left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right]z^3 + \dots
\end{aligned}$$

Thus,

Using (3.5) and (3.7) in (3.8), we obtain

$$\begin{aligned}
a_2 &= \frac{(A-B)c_1}{2(1+\alpha)}, \\
a_3 &= \frac{(A-B)}{8(1+\alpha)^2(1+2\alpha)} \left[2(1+\alpha)^2 c_2 + \left\{ (1+3\alpha)A - \left[(1+3\alpha) + (1+\alpha)^2 \right] B - (1+\alpha)^2 \right\} c_1^2 \right], \\
a_4 &= \frac{(A-B)}{48(1+\alpha)^3(1+2\alpha)(1+3\alpha)} \left[\begin{aligned} &8(1+\alpha)^3(1+2\alpha)c_3 + (1+\alpha)^2 \left\{ \begin{aligned} &6(1+5\alpha)A \\ &-2(7+27\alpha+8\alpha^2)B \\ &-8(1+\alpha)(1+2\alpha) \end{aligned} \right\} c_1 c_2 \\ &+ \left\{ \begin{aligned} &\left((1+6\alpha+17\alpha^2)A^2 + \left[\begin{aligned} &3(1+\alpha)^2(3+11\alpha+4\alpha^2) \\ &+ (1+6\alpha+17\alpha^2) \end{aligned} \right] B^2 \end{aligned} \right\} c_1^2 \\ &- \left[3(1+5\alpha)(3+8\alpha+\alpha^2) - 4(1+2\alpha)(1+7\alpha) \right] AB \\ &- 3(1+5\alpha)(1+\alpha)^2 A \\ &+ (1+\alpha)^2(7+27\alpha+8\alpha^2)B + 2(1+\alpha)^3(1+2\alpha) \end{aligned} \right\} c_1^3 \end{aligned} \right] \quad (3.9)
\end{aligned}$$

(3.9) yields,

$$a_2 a_4 - a_3^2 = \frac{(A-B)^2}{C(\alpha)} \left\{ Lc_1(4c_3) + Mc_1^2(2c_2) + Nc_1^4 - R(4c_2^2) \right\} \quad (3.10)$$

Where

$$C(\alpha) = 192(1+\alpha)^4(1+2\alpha)^2(1+3\alpha),$$

$$L = 4(1+\alpha)^3(1+2\alpha)^2,$$

$$\begin{aligned}
M &= 6\alpha(1+\alpha)^3 A + (1+\alpha)^2 \left[\begin{aligned} &12(1+3\alpha)(2+5\alpha+\alpha^2) \\ &-4(1+2\alpha)(7+27\alpha+8\alpha^2) \end{aligned} \right] B \\
&\quad - 2(1+\alpha)^3 [1+4\alpha+7\alpha^2],
\end{aligned}$$

$$\begin{aligned}
N = & [6(1+2\alpha)(1+3\alpha)(1+5\alpha) - 4(1+2\alpha)^2(1+7\alpha) - 3(1+3\alpha)^3]A^2 \\
& + \left[\begin{aligned} & 6(1+2\alpha)(1+5\alpha)(1+\alpha)^2 + 6(1+2\alpha)(1+3\alpha)(1+5\alpha) \\ & + 4(1+\alpha)^3(1+2\alpha)^2 - 4(1+7\alpha)(1+2\alpha)^2 - 3(1+3\alpha)^3 \\ & - 3(1+\alpha)^4(1+3\alpha) - 6(1+3\alpha)^2(1+\alpha)^2 \end{aligned} \right]B^2 \\
& + \left[\begin{aligned} & 6(1+3\alpha)^3 + 6(1+3\alpha)^2(1+\alpha)^2 - 12(1+5\alpha)(1+3\alpha)(1+2\alpha) \\ & - 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) + 8(1+2\alpha)^2(1+7\alpha) \end{aligned} \right]AB \\
& - 6\alpha(1+\alpha)^3A + 2(1+\alpha)^2 \left[\begin{aligned} & 4(1+2\alpha)^2(1+\alpha) + 3(1+2\alpha)(1+5\alpha) \\ & - 3(1+3\alpha)^2 - 3(1+\alpha)^2(1+3\alpha) \end{aligned} \right]B \\
& + (1+\alpha)^3[1+4\alpha+7\alpha^2]
\end{aligned}$$

And

$$R = 3(1+3\alpha)(1+\alpha)^4.$$

Using Lemma 2.1 and Lemma 2.2 in (3.10), we get

$$\begin{aligned}
\left| a_2 a_4 - a_3^2 \right| = & \frac{(A-B)^2}{C(\alpha)} \left[\begin{aligned} & \left\{ \begin{aligned} & 6(1+5\alpha)(1+3\alpha)(1+2\alpha) \\ & - 4(1+2\alpha)^2(1+7\alpha) - 3(1+3\alpha)^3 \end{aligned} \right\} A^2 \\ & + \left\{ \begin{aligned} & 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) \\ & + 6(1+5\alpha)(1+3\alpha)(1+2\alpha) \\ & + 4(1+\alpha)^3(1+2\alpha)^2 \\ & - 4(1+2\alpha)^2(1+7\alpha) - 3(1+3\alpha)^3 \\ & - 3(1+\alpha)^4(1+3\alpha) - 6(1+3\alpha)^2(1+\alpha)^2 \end{aligned} \right\} B^2 \\ & + \left\{ \begin{aligned} & 6(1+3\alpha)^3 + 6(1+3\alpha)^2(1+\alpha)^2 \\ & - 12(1+5\alpha)(1+3\alpha)(1+2\alpha) \\ & - 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) \\ & + 8(1+2\alpha)^2(1+7\alpha) \end{aligned} \right\} AB \\ & + \left[\begin{aligned} & 6(1+\alpha)^2 \{ (1+5\alpha)(1+2\alpha) - (1+3\alpha)^2 \} A \\ & + \left\{ \begin{aligned} & 6(1+5\alpha)(1+3\alpha)^2 \\ & + 6(1+\alpha)^4(1+3\alpha) \\ & - 8(1+\alpha)^3(1+2\alpha)^2 \\ & - 6(1+5\alpha)(1+\alpha)^2(1+2\alpha) \end{aligned} \right\} B \end{aligned} \right] c_1^2 x (4 - c_1^2) \\ & - \left[\begin{aligned} & 12(1+\alpha)^4(1+3\alpha) \\ & + \{ 4(1+\alpha)^3(1+2\alpha)^2 - 3(1+\alpha)^4(1+3\alpha) \} c_1^2 \end{aligned} \right] x^2 (4 - c_1^2) \\ & + 8(1+2\alpha)^2(1+\alpha)^3 c_1 (4 - c_1^2) (1 - |x|^2) \end{aligned} \right].
\end{aligned}$$

Assume that $c_1 = c$ and $c \in [0, 2]$, using triangular inequality and $|z| \leq 1$, we have

$$\left| a_2 a_4 - a_3^2 \right| \leq \frac{(A-B)^2}{C(\alpha)} + \left[\begin{aligned} & \left\{ \frac{6(1+5\alpha)(1+3\alpha)(1+2\alpha) - 4(1+2\alpha)^2(1+7\alpha)}{-3(1+3\alpha)^3} \right\} A^2 \\ & + \left\{ \begin{aligned} & \frac{6(1+5\alpha)(1+\alpha)^2(1+2\alpha)}{+6(1+5\alpha)(1+3\alpha)(1+2\alpha)} \\ & + \frac{4(1+\alpha)^3(1+2\alpha)^2}{-4(1+2\alpha)^2(1+7\alpha)-3(1+3\alpha)^3} \\ & - \frac{3(1+\alpha)^4(1+3\alpha)-6(1+3\alpha)^2(1+\alpha)^2}{-3(1+\alpha)^4(1+3\alpha)-6(1+3\alpha)^2(1+\alpha)^2} \end{aligned} \right\} B^2 \\ & + \left\{ \begin{aligned} & \frac{6(1+3\alpha)^3 + 6(1+3\alpha)^2(1+\alpha)^2}{-12(1+5\alpha)(1+3\alpha)(1+2\alpha)} \\ & - \frac{6(1+5\alpha)(1+\alpha)^2(1+2\alpha)}{+8(1+2\alpha)^2(1+7\alpha)} \end{aligned} \right\} AB \\ & + \left\{ \begin{aligned} & \frac{6(1+\alpha)^2 \{ (1+5\alpha)(1+2\alpha) - (1+3\alpha)^2 \}}{+6(1+5\alpha)(1+3\alpha)^2 + 6(1+\alpha)^4(1+3\alpha)} \\ & - \frac{8(1+\alpha)^3(1+2\alpha)^2}{-6(1+5\alpha)(1+\alpha)^2(1+2\alpha)} \end{aligned} \right\} B \\ & + \left[\begin{aligned} & 12(1+\alpha)^4(1+3\alpha) + \left\{ \frac{4(1+\alpha)^3(1+2\alpha)^2}{-3(1+\alpha)^4(1+3\alpha)} \right\} c^2 \\ & + 8(1+2\alpha)^2(1+\alpha)^3 c (4-c^2) (1-\delta^2) \end{aligned} \right] \delta^2 (4-c^2) \end{aligned} \right] c^4.$$

$$= \frac{(A-B)^2}{C(\alpha)} F(\delta), \text{ where } \delta = |x| \leq 1 \text{ and}$$

Consequently

$$\left| a_2 a_4 - a_3^2 \right| \leq \frac{(A-B)^2}{C(\alpha)} G(c), \quad (3.11)$$

Where

$$G(c) = F(1).$$

So

$$G(c) = -P(\alpha)c^4 + Q(\alpha)c^2 + 48(1+\alpha)^4(1+3\alpha),$$

Where $P(\alpha)$ and $Q(\alpha)$ are defined in (3.2) and (3.3) respectively.

Now

$$G'(c) = -4P(\alpha)c^3 + 2Q(\alpha)c$$

And

$$G''(c) = -12P(\alpha)c^2 + 2Q(\alpha).$$

$G'(c) = 0$ gives

$$c \{ -4P(\alpha)c^2 + 2Q(\alpha) \} = 0.$$

$G''(c)$ is negative at $c = \sqrt{\frac{Q(\alpha)}{2P(\alpha)}} = c'$.

$$F(\delta) = \left[\begin{aligned} & \left\{ \frac{6(1+5\alpha)(1+3\alpha)(1+2\alpha)}{-4(1+2\alpha)^2(1+7\alpha)-3(1+3\alpha)^3} \right\} A^2 \\ & + \left\{ \begin{aligned} & \frac{6(1+5\alpha)(1+\alpha)^2(1+2\alpha)}{+6(1+5\alpha)(1+3\alpha)(1+2\alpha)+4(1+\alpha)^3(1+2\alpha)^2} \\ & - \frac{4(1+2\alpha)^2(1+7\alpha)-3(1+3\alpha)^3}{-3(1+\alpha)^4(1+3\alpha)-6(1+3\alpha)^2(1+\alpha)^2} \end{aligned} \right\} B^2 \\ & + \left\{ \begin{aligned} & \frac{6(1+3\alpha)^3 + 6(1+3\alpha)^2(1+\alpha)^2}{-12(1+5\alpha)(1+3\alpha)(1+2\alpha)} \\ & - \frac{6(1+5\alpha)(1+\alpha)^2(1+2\alpha)+8(1+2\alpha)^2(1+7\alpha)}{+8(1+2\alpha)^2(1+7\alpha)} \end{aligned} \right\} AB \\ & + \left\{ \begin{aligned} & \frac{6(1+\alpha)^2 \{ (1+5\alpha)(1+2\alpha) - (1+3\alpha)^2 \}}{+6(1+5\alpha)(1+3\alpha)^2 + 6(1+\alpha)^4(1+3\alpha)} \\ & - \frac{8(1+\alpha)^3(1+2\alpha)^2}{-6(1+5\alpha)(1+\alpha)^2(1+2\alpha)} \end{aligned} \right\} B \\ & + \left[\begin{aligned} & 12(1+\alpha)^4(1+3\alpha) + \left\{ \frac{4(1+\alpha)^3(1+2\alpha)^2}{-3(1+\alpha)^4(1+3\alpha)} \right\} c^2 \\ & + 8(1+2\alpha)^2(1+\alpha)^3 c (4-c^2) (1-\delta^2) \end{aligned} \right] \delta^2 (4-c^2) \end{aligned} \right] c^4.$$

Is an increasing function. Therefore $\text{Max} F(\delta) = F(1)$.

So

$$\text{Max}G(c) = G(c').$$

Hence from (3.11), we obtain (3.1).

The result is sharp for $c_1 = c'$, $c_2 = c_1^2 - 2$ and $c_3 = c_1(c_1^2 - 3)$.

For $A=1$ and $B=-1$, Theorem 3.1 gives the following:

Corollary 3.1.1: If $f(z) \in M(\alpha)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2} \left[1 - \frac{12\alpha^2(1+\alpha)^4}{(1+3\alpha)A(\alpha)} \right],$$

Where

$$A(\alpha) = -24(1+2\alpha)(1+5\alpha)(2+5\alpha+\alpha^2) + 12(1+3\alpha)^2(3+7\alpha+2\alpha^2) - 4(1+\alpha)^3(1+4\alpha+7\alpha^2) + 16(1+2\alpha)^2(1+7\alpha).$$

For $\alpha=0$, Theorem 3.1 gives the following result due to Singh and Singh [16].

Corollary 3.1.2: If $f(z) \in S^*(A, B)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(A-B)^2}{4}.$$

For $\alpha=1$, Theorem 3.1 gives the following result due to Singh and Singh [16].

Corollary 3.1.3: If $f \in K(A, B)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(A-B)^2}{576} \left[\frac{16|A^2 + 2B^2 + AB| - |A-5B|^2 - 12|A-5B| - 36}{|-A^2 + 2B^2 + AB| - |A-5B| - 2} \right].$$

Putting $\alpha=0, A=1$ and $B=-1$ in Theorem 3.1, we obtain the following result due to Janteng et al. [5].

Corollary 3.1.4: If $f(z) \in S^*$, then

$$|a_2 a_4 - a_3^2| \leq 1.$$

Putting $\alpha=1, A=1$ and $B=-1$ in Theorem 3.1, we obtain the following result due to Janteng et al. [5].

Corollary 3.1.5: If $f(z) \in K$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

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