Butterfly Graphs with Shell Orders m and $2m+1$ are Graceful

Ezhilarasi Hilda Stanley and J. Jeba Jesintha

Abstract--- A graceful labelling of an un directed graph G with n edges is a one-one function from the set of vertices $V(G)$ to the set $\{0, 1, 2, \dots, n\}$ such that the induced edge labels are all distinct. An induced edge label is the absolute difference between the two end vertex labels. A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. A double shell is one vertex union of two shells. A bow graph is defined to be a double shell in which each shell has any order. In this paper we define a butterfly graph as a bow graph with exactly two pendant edges at the apex and we prove that all butterfly graphs with one shell of order m and the other shell of order $(2m+1)$ are graceful.

Keywords--- Bow Graph, Butterfly Graph, Graceful Labelling, Shell Graph

I. INTRODUCTION

IN 1967 Rosa [11] introduced the labelling method called β -valuation as a tool for decomposing the complete graph into isomorphic sub graphs. Later on, this β -valuation was re named as graceful labelling by Golomb [6]. A graceful labelling of a graph G with ' q ' edges and vertex set V is an injection $f: V(G) \rightarrow \{0, 1, 2, \dots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labelling is called a graceful graph. Various kinds of graphs are shown to be graceful. In particular, cycle-related graphs have been a major focus of attention for nearly five decades. Rosa[11] showed that the n -cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$. Frucht [4] has shown that the Wheels

$W_n = C_n + K_1$ are graceful. Helms H_n (graph obtained from a wheel by attaching a pendent edge at each vertex of the n -cycle) are shown to be graceful by Ayel and Favaron[1]. Koh, Rogers, Teo and Yap[9] defined a web graph as one obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. The web graph was proved to be graceful by Kang, Liang, Gao and Yang [8].

Delorme, Koh *et al* [3] showed that any cycle with a chord is graceful. In 1985 Koh, Rogers, Teo and Yap [10] defined a cycle with a P_k -chord to be a cycle with the path P_k joining two non consecutive vertices of the cycle and proved that these graphs are graceful when $k = 3$. For an exhaustive survey, refer to the dynamic survey by Gallian [5].

Deb and Limaye [2] have defined a shell graph as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n-3)$ see Fig.1.

Figure 1: Shell Graph $C(n, n-3)$

Note that the shell $C(n, n-3)$ is the same as the fan $F_{n-1} = P_{n-1} + K_1$. A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex. In [7] a bow graph is defined to be a double shell in which each shell has any order. In this paper we first define a Butterfly graph as a bow graph with exactly two pendant edges at the apex and we prove that all butterfly graphs with shells of order m and $(2m+1)$ excluding the apex are graceful.

II. MAIN RESULT

In this section we prove that butterfly graphs with shells of order m and $(2m+1)$ (order excludes the apex) are graceful.

Theorem: All butterfly graphs with shell orders m and $(2m+1)$ (Order excludes the apex) are graceful.

Proof: Let G be a butterfly graph with shells of order m and $(2m+1)$ excluding the apex. (Note that the shell order excludes the apex). Let the number of vertices in G be

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' n ' and the number of edges be ' q '. We describe the graph G as follows: In G , the shell that is present to the left of the apex is called as the left wing and the shell that is present to the right of the apex is considered as the right wing. Let m be the order of the right wing of G and $(2m + 1)$ is the order of the left wing of G . The apex of the butterfly graph is denoted as v_0 . Denote the vertices in the wing of the butterfly from bottom to top as $v_1, v_2, \dots, v_m$. The vertices in the left wing of the butterfly are denoted from top to bottom as $v_{m+1}, v_{m+2}, \dots, v_{3m}, v_{(3m+1)}$. The vertices in the pendant edges are $v_{(3m+2)}, v_{(3m+3)}$. (See fig. 2). Note that $n = (3m + 4)$ and $q = (6m + 2)$.

We label the vertices of the butterfly graph as shown in fig 2..

Case 1 : When m is odd.

Here $m = 2j + 3$ where $j = 0, 1, 2, 3, \dots$

Define

$$f(v_0) = 0 \quad \dots (1)$$

$$f(v_{2i-1}) = \begin{cases} 4m + 2j + 2i + 2, & \text{for } 1 \leq i \leq (m-j-1) \\ 6m + 2j - 2i + 7, & \text{for } (m-j) \leq i \leq (m+j+2) \\ 1, & \text{for } i = (m+j+3) \end{cases} \quad \dots (2)$$

$$f(v_{2i}) = \begin{cases} 4m + 2j - 2i + 4, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m - 2j + 2i - 4, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ q, & \text{for } i = (m+j+3) \end{cases} \quad \dots (3)$$

From the above definition given in (1), (2), (3) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$$|f(v_0) - f(v_{2i-1})| = \begin{cases} 4m + 2j + 2i + 2, & \text{for } 1 \leq i \leq (m-j-1) \\ 6m + 2j - 2i + 7, & \text{for } (m-j) \leq i \leq (m+j+2) \\ 1, & \text{for } i = (m+j+3) \end{cases} \quad \dots (4)$$

$$|f(v_0) - f(v_{2i})| = \begin{cases} 4m + 2j - 2i + 4, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m - 2j + 2i - 4, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ q, & \text{for } i = (m+j+3) \end{cases} \quad \dots (5)$$

$$\begin{cases} 4i-2, & \text{for } 1 \leq i \leq (m-j-2) \end{cases}$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} 4m + 4j - 4i + 11, & \text{for } (m-j) \leq i \leq (m+j+2) \\ \dots \end{cases} \quad (6)$$

Figure 2: A Butterfly Graph with $n=(3m+4)$ Vertices

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} 4i, & \text{for } 1 \leq i \leq (m-j-2) \\ 4m + 4j - 4i + 9, & \text{for } (m-j-1) \leq i \leq (m+j+1) \\ \dots \end{cases} \quad (7)$$

From the computations given in (4), (5), (6), (7) we can see that the edge labels are distinct.

Case 2: When m is even.

Here $m = 2j + 4$, where $j = 0, 1, 2, 3, \dots$

Define

$$f(v_0) = 0 \quad \dots (8)$$

$$f(v_{2i-1}) = \begin{cases} 4m + 2j - 2i + 6, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m - 2j + 2i - 6, & \text{for } (m-j-1) \leq i \leq (m+j+3) \\ q, & \text{for } i = (m+j+4) \end{cases} \quad \dots (9)$$

$$f(v_{2i}) = \begin{cases} 4m + 2j + 2i + 4, & \text{for } 1 \leq i \leq (m-j-2) \\ 6m + 2j - 2i + 7, & \text{for } (m-j-1) \leq i \leq (m+j+3) \\ 1, & \text{for } i = (m+j+3) \end{cases} \quad \dots (10)$$

From the above definition given in (8), (9), (10) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$$|f(v_0) - f(v_{2i-1})| = \begin{cases} 4m + 2j - 2i + 6, & \text{for } 1 \leq i \leq (m-j-2) \\ 2m - 2j + 2i - 6, & \text{for } (m-j-1) \leq i \leq (m+j+3) \\ q, & \text{for } i = (m+j+4) \\ \dots \end{cases} \quad (11)$$

$$|f(v_0) - f(v_{2i})| = \begin{cases} 4m + 2j + 2i + 4, & \text{for } 1 \leq i \leq (m-j-2) \\ 6m + 2j - 2i + 7, & \text{for } (m-j-1) \leq i \leq (m+j+3) \\ 1, & \text{for } i = (m+j+3) \\ \dots \end{cases} \quad (12)$$

$$|f(v_{2i-1}) - f(v_{2i})| = \begin{cases} 4i - 2, & \text{for } 1 \leq i \leq (m-j-2) \\ 4m + 4j - 4i + 13, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ \dots \end{cases} \quad (13)$$

$$|f(v_{2i}) - f(v_{2i+1})| = \begin{cases} 4i, & \text{for } 1 \leq i \leq (m-j-3) \\ 4m + 4j - 4i + 11, & \text{for } (m-j-1) \leq i \leq (m+j+2) \\ \dots \end{cases} \quad (14)$$

From the computation given in (11), (12), (13), (14) it is clear that the edge labels are distinct. Hence butterfly

graphs with shells of order m and $(2m + 1)$ excluding the apex are graceful.

III. CONCLUSION

This paper presents the gracefulness of butterfly graphs with shells of order m and $(2m + 1)$. We look to obtain even greater results in the future related to one vertex union of many shell graphs.

APPENDIX

We illustrate both the cases as given below. Figure A depicts the case when 'm' is odd and figure B, when 'm' is

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Figure 3: Graceful Butterfly Graph with $m=7, n=25, q=44$

Figure 4: Graceful Butterfly Graph with $m=10, n=34, q=62$